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Virtual roots of real polynomials

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Abstract

The fact that a real univariate polynomial misses some real roots is usually overcome by considering complex roots, but the price to pay for, is a complete loss of the sign structure that a set of real roots is endowed with (mutual position on the line, signs of the derivatives, etc.). In this paper we present real substitutes for these missing roots which keep sign properties and which extend of course the existing roots. Moreover these "virtual roots" are the values of semialgebraic continuous – rather uniformly – functions defined on the set of monic polynomials. We present some applications. © 1998 Elsevier Science B.V.

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0. Introduction

The problem known as Pierce–Birkhoff conjecture is the following: take a real valued continuous function on $\mathbb{R}^n$ which is a piecewise polynomial, with a finite number of pieces (i.e. a "C_0-spline"); can you write it down as a finite combination of sup and inf of polynomials? Under this form the problem has been solved for $n \leq 2$ and the proof for $n = 2$ [3] (see also [2]) uses actually a certain parametrization of the

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one-dimensional case. Unfortunately this parametrization is not good enough to get the result for higher dimension and one is still looking for some path in this direction.

Actually, the proof in the low dimension case uses the notion of "truncation of a polynomial" which is the following: if u is the rth real zero of the degree d univariate polynomial $P(X)$, the "rth truncation" of P, $\phi_{d,r}(P)$ is the function defined as 0 when $x \leq u$ and equal to $P(x)$ for $x \geq u$. The essential point is that $\phi_{d,r}(P)$ is an Inf–Sup definable function (ISD in short) and that its formal description with sup and inf is the same for every other polynomial Q as long as the relative position of the real roots of all the successive derivatives of Q is the same as for P. This kind of "local uniformity" makes possible to define $\phi_{d,r}(P)$ for multivariate polynomials $P(\underline{X}, Y)$, considering $\underline{X}$ as parameters, as long as $\underline{X}$ belongs to some semi-algebraic set, precisely described by the sign conditions which define the position of the zeroes of the Y-derivatives of P; and this is sufficient to get the proof in dimension 2. But, for higher dimensions, we need more uniformity in the one-dimensional case. In particular, it would be nice to have this partially defined function $\phi_{d,r}$, defined everywhere on the parameter space. This is of course impossible in general: the rth real zero alone need not exist for a given value of $\underline{X}$. Of course, life would be easier if every monic degree d polynomial would have d real roots!

Actually, the notions of "virtual root" we are going to introduce in this paper will give a good substitute to this unreachable paradise and will, in some sense, "render hyperbolic every polynomial" (a polynomial is hyperbolic when its roots are real). More precisely, we have two classes of "virtual root functions" defined on the set of degree d monic univariate polynomials of $\mathbb{R}[Y]$ (which can be identified to $\mathbb{R}^d$) and one of these classes is the following:

> For every integer $d \geq 1$ and every integer $0 < j \leq d$, there is a real valued semi-algebraic continuous function $\rho_{d,j}$ on $\mathbb{R}^d$, such that $\rho_{d,j}(P)$ is the jth real root of P when P is hyperbolic, and which satifies in addition the sign conditions we expect for an actual jth root. For example $\rho_{d,j}(P) \leq \rho_{d-1,j}(P') \leq \rho_{d,j+1}(P)$ if P' is the derivative of P.

Then, once we have our hands on the rth virtual root $\rho_{d,r}(P)(\underline{X})$ of a degree $d \geq r$ monic polynomial $P(\underline{X}, Y)$ everywhere on the parameter space, the next step towards a solution of Pierce–Birkhoff conjecture would be to construct the "rth virtual truncation" of P as an ISD function coinciding with P for $Y \geq \rho_{d,r}$ and "going to zero as fast as possible" for $Y \leq \rho_{d,r}$, and giving of course the actual truncation in case $\rho_{d,r}$ is an actual root. This is not yet completely worked out but should be available in the near future. Nevertheless, as early applications of these notions, we prove here the two following results:

(1) a continuous version of Thom's lemma,

(2) the closure under Sup and Inf of the ring generated by the virtual roots is the integral closure of the polynomial ring $\mathbb{R}[X_1, \ldots, X_n]$ inside the real valued continuous functions on $\mathbb{R}^n$.

The paper is organized under the following headings:
1. General tools
2. The rth virtual root
3. Thom's virtual roots
4. Examples
5. Links and common properties
6. Applications

In what follows, we have chosen to work with the real numbers $\mathbb{R}$, but the discussion is valid for any real closed field. Furthermore, if the polynomials we start with have their coefficients in a subfield $\mathbf{K}$ of a real closed field $\mathbf{R}$, every new constructed polynomial also has its coefficients in this field $\mathbf{K}$.

1. General tools

As we said in the abstract, we want to define on the basis of the set of monic univariate real polynomials, some collections of "virtual root" functions, extending everywhere the actual root functions in such a way that some sign conditions are preserved. There are essentially two ways to distinguish a given real root of a polynomial from the others: one is the rank of this root, the other is the collection of the signs taken at this root by the derivatives.

The main idea is the following simple observation: suppose P is a parametrized polynomial in one variable and we are following some particular real root along the parameters. If for some value of the parameters this root disappears, then it becomes a root of the derivative and this becomes our "virtual root". But in both cases, actual or virtual root, the root realizes the local minimum of the absolute value of P, and this is the key observation.

So, we are going to consider two sets of such root functions, called, respectively, "rth virtual root" and "Thom's virtual roots". In the first case we want to preserve the rank of a given root among the others. In the second case we try to preserve the sign that every derivative of P takes on a given root, but it is a bit more complicated.

The main tool to define these functions is the following one:

Definition 1.1. We identify the set of monic degree d polynomials of $\mathbf{R}[X]$ to $\mathbb{R}^d$, and P will be understood as a polynomial or as a point in $\mathbb{R}^d$ as well. Let $\mathscr{S}_d$ be the closed $\mathbb{Q}$-semialgebraic set defined by:

$$\mathscr{S}_d = \{(a,b,P) : a \leq b,\ \deg(P) = d,\ \forall x, y \in [a,b] \quad P'(x)P'(y) \geq 0\}$$

and $\mathscr{R}_d$ be the semi-algebraic function defined on $\mathscr{S}_d$ by

$$\mathscr{R}_d(a,b,P) = z \quad \text{such that} \quad |P(z)| = \min\{|P(u)| : u \in [a,b]\}.$$

An easy verification shows that the function $\mathscr{R}_d$ satisfies the following equality:

$$\mathscr{R}_d(a,b,P) = \begin{cases} a = b & \text{if } a = b, \\ a & \text{if } (P(b)-P(a))P(a) \geq 0, \\ b & \text{if } (P(b)-P(a))P(b) \leq 0, \\ \text{the real root of } P \text{ in } (a,b) & \text{otherwise.} \end{cases} \qquad (\star)$$

Proposition 1.2. *If d is a nonnegative integer then*:

(1) *if $(a,b,P) \in \mathscr{S}_d$ and P has a real root z on $[a,b]$ then $\mathscr{R}_d(a,b,P) = z$,*

(2) *the function $\mathscr{R}_d$ is continuous on $\mathscr{S}_d$,*

(3) *if $(a,b,P) \in \mathscr{S}_d$ then the number $x = \mathscr{R}_d(a,b,P)$ can be characterized by the following inequalities*:

$$\begin{aligned} & a \leq x \leq b, \\ & (x-a)P(a)(P(b)-P(a)) \leq 0, \qquad (x-a)P(x)(P(b)-P(a)) \leq 0, \\ & (b-x)P(b)(P(b)-P(a)) \geq 0, \qquad (b-x)P(x)(P(b)-P(a)) \geq 0. \end{aligned}$$

Proof. Parts (1) and (3) are easy considering the different cases appearing in the formula $(\star)$. Next we prove part (2) which is no more than proving that the real root of a monotone polynomial in an interval varies continuously with the coefficients. Let (a,b,P) be an element in $\mathscr{S}_d$ and ε a strictly positive element of $\mathbb{R}$. We search for a δ giving the continuity of the function $\mathscr{R}_d$.

If $b - a \leq \varepsilon/2$ then taking $\delta = \varepsilon/2$ we have

$$\begin{aligned} |a-a'| + |b-b'| + |P-R| < \delta \implies |x-x'| &\leq \max\{b,b'\} - \min\{a,a'\} \\ &\leq |b-a| + |a-a'| + |b-b'| < \varepsilon \end{aligned}$$

with $(a',b',R) \in \mathscr{S}_d$, $x = \mathscr{R}_d(a,b,P)$ and $x' = \mathscr{R}_d(a',b',R)$.

If $b-a \geq \varepsilon/2$ and $x = \mathscr{R}_d(a,b,P)$ then, writing α for $+1$ or -1 according to the sign of $P(b) - P(a)$, we consider three cases:

- If $x < a + \varepsilon/2$ then $\alpha \cdot P(a+\varepsilon/2) > 0$. For a sufficiently small variation δ of (a,b,P) in $\mathscr{S}_d$, the real number $\alpha \cdot P(a+\varepsilon/2)$ remains strictly positive, the variation of a is smaller than $\varepsilon/2$ and $\mathscr{R}_d(a,b,P)$ remains on the interval $[a, a+\varepsilon/2)$.
- If $x > b - \varepsilon/2$, we proceed the same way as in the previous case.
- If $a + \varepsilon/2 \leq x \leq b - \varepsilon/2$, then $\alpha \cdot P(x - \varepsilon/4) < 0 < \alpha \cdot P(x + \varepsilon/4)$. For a sufficiently small variation δ of (a,b,P) in $\mathscr{S}_d$, $\alpha \cdot P(x-\varepsilon/4)$ remains < 0, $\alpha \cdot P(x+\varepsilon/4)$ remains strictly positive, and the variations of a and b are smaller than $\varepsilon/4$. So, $\mathscr{R}_d(a,b,P)$ remains in the open interval $(x - \varepsilon/4, x + \varepsilon/4)$. □

Next we generalize the definition of $\mathscr{R}_d$ to the cases $a = -\infty$ or $b = +\infty$. This is achieved by considering the semialgebraic sets:

$$\begin{aligned} & \mathscr{S}_{d,+} = \{(a,P)\colon\ \forall x \in [a,+\infty)\ P'(x) \geq 0\}, \\ & \mathscr{S}_{d,-} = \{(b,P)\colon\ \forall x \in (-\infty,b]\ (-1)^{d-1}P'(x) \geq 0\} \end{aligned}$$

defining on $\mathscr{S}_{d,+}$:

$$\mathscr{R}_d(a,+\infty,P)=\mathscr{R}_d\left(a,\max\left(a,1+\sup_{i=0,\dots,d-1}\{|a_i|\}\right),P\right)$$

and defining on $\mathscr{S}_{d,-}$:

$$\mathscr{R}_d(-\infty,b,P)=\mathscr{R}_d\left(\min\left(b,-1-\sup_{i=0,\dots,d-1}\{|a_i|\}\right),b,P\right).$$

Notation 1.3. If Q is a univariate polynomial then $Q^{(i)}$ will denote the ith derivative of Q, with $Q^{(0)}=Q$, $\deg(Q)$ will be the degree of Q and $\mathrm{lcof}(Q)$ its leading coefficient. In order to be able to use the identification between $\mathbb{R}^d$ and the set of monic degree d polynomials, we define:

$$Q^{[i]}=\frac{Q^{(d-i)}}{\mathrm{lcof}(Q^{(d-i)})}$$

as the normalized derivative of Q of degree i. We define Q^* as the product of all normalized derivatives $Q^{[i]}$ of Q (Q included).

2. The rth virtual root

Let $P\in\mathbb{R}[X]$ be a monic degree d polynomial. For every integer r such that $0<r\leq d$, we want to define a function $\rho_{d,r}$ on $\mathbb{R}^d$ having the following properties:

(1) $\rho_{d,r}$ is a continuous semi-algebraic function on $\mathbb{R}^d$,

(2) if P is hyperbolic and $u\in\mathbb{R}$ is the rth real root of the polynomial P, then $u=\rho_{d,r}(P)$,

(3) $\rho_{d,r}(P)\leq\rho_{d-1,r}(P'/d)\leq\rho_{d,r+1}(P)$.

The restriction to monic polynomials is not really essential: we could be satisfied with polynomials such that the leading coefficient never vanishes, but then we would loose some uniformity in the continuity of $\rho_{d,r}$ (see Section 5). But without loss of generality, we may as well replace monic by "quasi-monic", meaning that the leading coefficients are at least 1. Anyway, for simplicity, we will do everything with monic polynomials.

Definition 2.1. Let $P(x)=x^d-(a_{d-1}x^{d-1}+\cdots+a_0)$ be a monic polynomial in $\mathbb{R}[x]$. For $d\geq 0$ and for any integer j, we define $\rho_{d,j}(P)$ in the following inductive way:

- if $j\leq 0$, we put $\rho_{d,j}(P)=-\infty$,
- if $j>d$, we put $\rho_{d,j}(P)=\infty$,
- if $d>0$ and $1\leq j\leq d$, we define

$$\rho_{d,j}(P)\overset{\text{def}}{=}\mathscr{R}_d(\rho_{d-1,j-1}(P'/d),\ \rho_{d-1,j}(P'/d),P).$$

In particular, if $P=X-a$ then $\rho_{1,1}(P)=a$. Let us also define the sets:

$$U_{d,j}(P)\overset{\text{def}}{=}\{\alpha\in\mathbb{R}:\rho_{d,j-1}(P)<\alpha<\rho_{d,j}(P)\}.$$

(These sets are open intervals when they are not empty, and they are empty in particular for $j < 0$ and $j > d$.) For simplicity, we will often write $U_{d,j}(P')$ and $\rho_{d,j}(P')$ instead of the corresponding terms with P'/d. Applying Proposition 1.2, it is easy to prove by induction the following proposition.

Proposition 2.2. *For $d > 0$ and $0 < j \leq d$, the functions $\rho_{d,j}$ are integral continuous functions on $\mathbb{R}^d$ defined over $\mathbb{Q}$ and they are roots of the polynomial P^*. On the other hand, every root of P is equal to some $\rho_{d,j}(P)$.*

Let us quote here the basic properties of these $\rho_{d,r}$ which make P appear like hyperbolic with respect to the virtual roots:

Proposition 2.3. *For $d > 0$, the functions $\rho_{d,r}$ have the following properties:*
(1) $\forall r\ \rho_{d,r}(P) \leq \rho_{d-1,r}(P') \leq \rho_{d,r+1}(P)$;
(2) *Every monic degree d polynomial has d virtual roots (possibly equal)*;
(3) $(-1)^{d+r}P(x) > 0$ *for* $x \in U_{d,r+1}(P)$.

Proof. Parts (1) and (2) are just from the definition. For (3), we make an induction on d. Anyway, there is something to prove only when the interval $(\rho_{d,r}(P), \rho_{d,r+1}(P))$ is not empty, so we may assume $0 \leq r \leq d$.

If $d = 1$, it is easily checked. If $d > 1$, we have

$$\rho_{d-1,r-1}(P') \leq \rho_{d,r}(P) \leq \rho_{d-1,r}(P') \leq \rho_{d,r+1}(P) \leq \rho_{d-1,r+1}(P').$$

we consider two cases:
- if $\rho_{d,r}(P) = \rho_{d-1,r}(P')$, then $U_{d,r+1}(P) \subseteq U_{d-1,r+1}(P')$ and we know by hypothesis that $(-1)^{d+r}P' < 0$ on $U_{d-1,r+1}(P')$. So $(-1)^{d+r}P$ is decreasing on $U_{d,r+1}(P)$. As $\rho_{d,r+1}(P)$ realizes the minimum of $|P|$ on $U_{d-1,r+1}(P')$, we get that $(-1)^{d+r}P > 0$ on $U_{d,r+1}(P)$.
- If $\rho_{d,r}(P) < \rho_{d-1,r}(P')$, then $(-1)^{d-1+r-1}P' > 0$ on $U_{d-1,r}(P')$ and $(-1)^{d+r}P$ is increasing on this interval. As $\rho_{d,r}(P)$ realizes the minimum of $|P|$ on this interval, $(-1)^{d+r}P$ must be positive on $U_{d-1,r}(P') \cap U_{d,r+1}(P) \neq \emptyset$, and must be so on the whole of $U_{d,r+1}(P)$, for it cannot change sign on this interval. $\square$

3. Thom's virtual roots

Let $P(x) = x^d - (a_{d-1}x^{d-1} + \cdots + a_0)$ be a monic polynomial in $\mathbb{R}[x]$. Thom's lemma says in particular that if we fix the sign (in the large sense) of every derivative of P, we get a set containing at most one root of P. The virtual roots we are going to build up are real numbers x_σ indexed by a list of signs $\sigma = [\sigma_0, \ldots, \sigma_{d-1}]$ having the property that when the $d-1$ nontrivial derivatives of P take the sign given by the list σ at some real root of P, then x_σ is precisely this root. Of course, it may happen that the sign conditions on the derivatives produce an empty set: in that case the point x_σ cannot satisfy the sign conditions (although it might be an actual root of P).

Notation 3.1. We shall denote by $\sigma = [\sigma_0, \dots, \sigma_d]$ a list of signs, $\sigma_i \in \{+,-\}$. The "length" $\lg(\sigma)$ will be d and σ_0 will always be equal to $+$. The convenience of this σ_0 will appear later. Concerning the list σ we introduce the following symbols, for $i = 1, \dots, d$:

$$\widetilde{\sigma}_i = \begin{cases} > & \text{if } \sigma_i = +, \\ < & \text{if } \sigma_i = -, \end{cases} \qquad \overline{\sigma}_i = \begin{cases} \geq & \text{if } \sigma_i = +, \\ \leq & \text{if } \sigma_i = -, \end{cases}$$

$$\sigma^{(i)} = [\sigma_0, \dots, \sigma_{d-i}], \qquad \sigma^{[i]} = [\sigma_0, \dots, \sigma_i].$$

The basic semi-algebraic open set:

$$\{\alpha \in \mathbb{R} : P^{[1]}(\alpha)\, \widetilde{\sigma}_1\, 0, \dots, P^{[d]}(\alpha)\, \widetilde{\sigma}_d\, 0\}$$

will be denoted by $U_\sigma(P)$ and the basic semi-algebraic closed set:

$$\{\alpha \in \mathbb{R} : P^{[1]}(\alpha)\, \overline{\sigma}_1\, 0, \dots, P^{[d]}(\alpha)\, \overline{\sigma}_d\, 0\}$$

by $F_\sigma(P)$.

With the previous notations Thom's Lemma (see [1] for a proof) can be stated in the following terms.

Theorem 3.2 (Thom's lemma). *If the closed set $F_\sigma(P)$ is not empty then it is a closed interval or a point, and its interior is always $U_\sigma(P)$. Moreover, every finite endpoint of the interval $F_\sigma(P)$ is a root of some $P^{(j)}$.*

Definition 3.3. Suppose $\deg P = \lg(\sigma) = d$ and $\varepsilon \in \{+,-\}$. Here we assume that $F_\sigma(P)$ is not empty. The two endpoints of $F_\sigma(P)$ will be denoted by: $\tau_\sigma^\varepsilon(P)$ with $\varepsilon = +$ for the right endpoint and $\varepsilon = -$ for the left endpoint.

There are two special cases where the interval $F_\sigma(P)$ is never empty and one of its endpoints is infinity:

$$\sigma = [+,+,+,\dots,+], \qquad \varepsilon = + \implies \tau_\sigma^\varepsilon(P) = +\infty,$$
$$\sigma = [+,-,+,-,+,\dots], \qquad \varepsilon = - \implies \tau_\sigma^\varepsilon(P) = -\infty.$$

Excepting the two infinity cases, the symbols τ_σ^ε represent semi-algebraic functions partially defined on $\mathbb{R}^d$. In the following two cases, the symbol τ_σ^ε provides a semi-algebraic function defined on the whole $\mathbb{R}^d$:

$$\sigma = [+,+,+,\dots,+], \qquad \varepsilon = - \implies \tau_\sigma^\varepsilon(P) = \max\{\alpha \in \mathbb{R} : P^*(\alpha) = 0\},$$
$$\sigma = [+,-,+,-,+,\dots], \qquad \varepsilon = + \implies \tau_\sigma^\varepsilon(P) = \min\{\alpha \in \mathbb{R} : P^*(\alpha) = 0\}.$$

Let us introduce the function, also partially defined on $\mathbb{R}^d$, denoted by $\rho_{\sigma^{(1)}}(P)$ and called *actual Thom's root* which is defined as the only real root of P inside the closed interval $[\tau^-_{[\sigma_0,\dots,\sigma_{d-1}]}(P'/d), \tau^+_{[\sigma_0,\dots,\sigma_{d-1}]}(P'/d)]$ when the endpoints of the interval are defined (possibly equal) and when such a root exists. The function $\rho_{\sigma^{(1)}}$, when defined, verifies the following equality:

$$\rho_{[\sigma_0,\dots,\sigma_{d-1}]}(P) = \tau^-_{[\sigma_0,\dots,\sigma_{d-1},\sigma_{d-1}]}(P) = \tau^+_{[\sigma_0,\dots,\sigma_{d-1},-\sigma_{d-1}]}(P).$$

It is clear from the definition that every root of P in $\mathbb{R}$, can be represented by some of these symbols. The functions τ and ρ will be extended as semialgebraic continuous functions to the whole of $\mathbb{R}^d$ in an inductive way.

Definition 3.4. If the degree d of P is equal to 1, we define

$$\rho_{[+]}(x-a) \stackrel{\text{def}}{=} \tau^+_{[+,-]}(x-a) \stackrel{\text{def}}{=} \tau^-_{[+,+]}(x-a) \stackrel{\text{def}}{=} a,$$
$$\tau^-_{[+,-]}(x-a) \stackrel{\text{def}}{=} -\infty, \qquad \tau^+_{[+,+]}(x-a) \stackrel{\text{def}}{=} +\infty.$$

If all the functions ρ and τ for degree $d-1$ are known then the definitions for degree d are:

$$\begin{aligned}\rho_{[\sigma_0,\dots,\sigma_{d-1}]}(P) &\stackrel{\text{def}}{=} \tau^-_{[\sigma_0,\dots,\sigma_{d-1},\sigma_{d-1}]}(P) \stackrel{\text{def}}{=} \tau^+_{[\sigma_0,\dots,\sigma_{d-1},-\sigma_{d-1}]}(P)\\ &\stackrel{\text{def}}{=} \mathscr{R}_d(\tau^-_{[\sigma_0,\dots,\sigma_{d-1}]}(P'/d),\ \tau^+_{[\sigma_0,\dots,\sigma_{d-1}]}(P'/d), P),\end{aligned}$$

$$\tau^+_{[\sigma_0,\dots,\sigma_{d-1},\sigma_{d-1}]}(P) \stackrel{\text{def}}{=} \tau^+_{[\sigma_0,\dots,\sigma_{d-1}]}(P'/d), \qquad \tau^-_{[\sigma_0,\dots,\sigma_{d-1},-\sigma_{d-1}]}(P) \stackrel{\text{def}}{=} \tau^-_{[\sigma_0,\dots,\sigma_{d-1}]}(P'/d).$$

Remark that if $\varepsilon \cdot \sigma_d = +$ then $\tau^\varepsilon_\sigma(P) = \tau^\varepsilon_{\sigma^{(1)}}(P'/d)$ and if $\varepsilon \cdot \sigma_d = -$ then $\tau^\varepsilon_\sigma(P) = \rho_{\sigma^{(1)}}(P)$. So we see inductively that excepting the infinity cases each function $P \mapsto \tau^\varepsilon_\sigma(P)$ is equal to some function $P \mapsto \rho_{\sigma^{[j]}}(P^{[j+1]})$, where $j < d$ depends only on σ and ε. We then get the following proposition.

Proposition 3.5. (1) *The above defined functions ρ_σ and τ^ε_σ are defined on the whole of $\mathbb{R}^d$ and are extensions of the partial functions introduced in Definition* 3.3.

(2) *The functions ρ_σ are integral, $\mathbb{Q}$-semi-algebraic and continuous on $\mathbb{R}^d$, and verify, for every monic polynomial P of degree d, the equality $P^*(\rho_\sigma(P)) = 0$.*

Proof. The proof is easy by induction on the degree, using Proposition 1.2 for the continuity and that $P^*(\rho_\sigma(P)) = 0$ to show it is integral in (2). □

In order to understand these functions ρ_σ, it is convenient to introduce the following definition.

Definition 3.6. For every monic polynomial P and every σ of length d, we define $G_\sigma(P) = [\tau^-_\sigma(P), \tau^+_\sigma(P)]$. By construction, this closed interval (may be a point) depends continuously on P and coincides with F_σ when the latter is not empty.

The main properties of G_σ are summarized below.

Proposition 3.7. *The interval G_σ has the following properties*:
(a) $G_{[+,+]}(x-a)=[a,+\infty]$ *and* $G_{[+,-]}(x-a)=[-\infty,a]$,
(b) *the interval* $G_{[\sigma_0,\ldots,\sigma_{d-1}]}(P')$ *is the union of the two intervals*

$$G_{[\sigma_0,\ldots,\sigma_{d-1},-\sigma_{d-1}]}(P) \quad \text{and} \quad G_{[\sigma_0,\ldots,\sigma_{d-1},\sigma_{d-1}]}(P)$$

with the right endpoint of the first one equal to the left endpoint of the second one,
(c) *if* $G_\sigma(P)$ *is not reduced to a point then* $G_\sigma(P)=F_\sigma(P)$.

Proof. Point (b) comes right from the Definition 3.4. For (c), it is clear by definition and case examination, that if $G_\sigma(P)$ is not reduced to a point then $G_\sigma(P)$ is the set of points α in $G_{\sigma^{(1)}}(P')$ such that $P(\alpha)\overline{\sigma}_d 0$. But then by induction on d, we may assume $G_{\sigma^{(1)}}(P')=\mathrm{F}_{\sigma^{(1)}}(P')$ and so $G_\sigma(P)=\mathrm{F}_\sigma(P)$. □

We can now understand better the functions ρ_σ themselves: the virtual Thom's roots are actual Thom's roots of some derivative:

Proposition 3.8. *For every monic polynomial P (resp. Q) of degree d (resp. $d+1$) and σ of length $d-1$ (resp. d), each $\rho_\sigma(P)$ (resp. $\tau_\sigma^\varepsilon(Q)$) is equal to an actual Thom's root of some $\rho_{\sigma^{[r-1]}}(P^{[r]})$.*

Proof. By definition of τ, it is sufficient to do it for ρ_σ. If $d=1$, it is the definition. If $d>1$, let $u=\rho_\sigma(P)$. By construction, $u\in G_\sigma(P^{[d-1]})$ and if u is not an endpoint of this interval, it is a root of P. But in that case, by Proposition 3.7(c), it is the actual Thom's root of P coded by σ. So we may assume u is an endpoint of this interval. Let r be the smallest integer such that u is an endpoint of $G_{\sigma^{[r+1]}}(P^{[r]})$. By Proposition 3.7(b), u is inside $G_{\sigma^{[r]}}(P^{[r-1]})$, and by the same argument as above, must be a root of $P^{[r]}$, coded by $\sigma^{[r]}$. □

In the case of rth virtual root the general pattern is quite easy: there is at most d virtual roots of degree d, naturally ordered and there is generically exactly d such distinct virtual roots (realized in specializing to hyperbolic polynomials). On the contrary, the situation for virtual Thom's roots is not so clear: How many such generic roots do we have and how are they mutually ordered? Is there some specialization that gives the 2^{d-1} a priori possible ρ_σ? We have the two following propositions.

Proposition 3.9. *For every $d\geq 1$ and every σ of length $d-1$, there is a real polynomial P of degree d such that $\rho_\sigma(P)$ is an actual Thom's root of P.*

Proof. It is sufficient to show that for any sign condition σ of length $d-1$ there is a real polynomial P of degree d having a root inside $U_\sigma(P)$. For degree 1 there is nothing to do and if $d>1$, by induction we may assume that there exists a Q of degree $d-1$ having a root in $U_{\sigma^{(1)}}(Q)$, making $U_\sigma(P)\neq\emptyset$ for any antiderivative P

of Q. Adjusting the constant term, it is then easy to find such a P having a root in $U_\sigma(P)$. □

Of course, this implies that there are 2^{d-1} distinct generic Thom's virtual roots.

Proposition 3.10. *Let* $s(d) = 1 + d(d-1)/2$.

(a) *Every monic degree d polynomial has at most $s(d)$ distinct Thom's virtual roots.*

(b) *For every $d \geq 1$ there exists a monic polynomial P which has $s(d)$ distinct Thom's virtual roots.*

Proof. By definition of $\rho_\sigma(P)$ (length$(\sigma) = d-1$), there is exactly one ρ_σ in each $G_\sigma(P')$, and in particular there is also exactly one in each nonempty F_σ. But the nonempty F_σ make a partition of $\mathbb{R}$ and their endpoints are zeroes of P'^*: the number of intervals is then bounded by one more than the number of roots of P'^*, which gives (a).

Now we show that this bound $s(d)$ for nonempty $F_\sigma(P)$ is effectively obtained. Choose P such that P^* is hyperbolic without multiple roots. If $d = 1$ or 2, it is clear. If $d > 2$, assume the number of intervals for P' (determined by sign conditions on $P^{(i)}$, $i \geq 2$) is $s(d-1)$, then there are $d-1$ intervals actually cut into two by the $d-1$ roots of P' and the number of nonempty F_σ is exactly $s(d-1) + d - 1 = s(d)$. Let us show that the ρ_σ corresponding to these $s(d)$ intervals produce $s(d)$ different real numbers: if two such ρ_σ would be equal, they would correspond to a common end of two consecutive intervals, realizing the minimal of the absolute value of P on the union of these two intervals. We have two cases:

(1) $|P|$ has a positive minimum at that point, but then cannot be hyperbolic,

(2) The point is a root of P, but is also a root of P'^* as an end of a F_σ, giving a double root to P^*: contradiction. □

Information about how the functions ρ_σ are ordered is summarized in the next proposition.

Proposition 3.11. *Assume that* $\deg(P) = d$ *and σ is a list of signs ($+$ or $-$) with* $\lg(\sigma) = d - 1$.

(a) *If μ is a list with length $k-1$ (with $d \geq k \geq 1$) different from σ, then the comparison, by $\geq$ or $\leq$, between $\rho_\sigma(P)$ and $\rho_\mu(P^{[k]})$ is given by the following rule involving only the signs in σ and μ:*

> *If i is the first index such that $\sigma_i \neq \mu_i$ (if μ is an initial segment of σ then $i = \lg(\mu) + 1$) then the sign of $\rho_\sigma(P) - \rho_\mu(P^{[k]})$ is equal to $\sigma_{i-1} \cdot \sigma_i$.*

(b) *If u is an element of $\mathbb{R}$ then the comparison between u and $\rho_\sigma(P)$ is given by "the same" rule than in* (a) *using the sign of $P^{[j]}(u)$ instead μ_j:*

> *If i is the first index such that $\sigma_i \neq \mathrm{sign}(P^{[i]}(u))$ and $i < d$ then the sign of $\rho_\sigma(P) - u$ is equal to $\sigma_{i-1} \cdot \sigma_i$. If $\sigma_i = \mathrm{sign}(P^{[i]}(u))$ for $i = 1, \ldots, d-1$ then the sign of $\rho_\sigma(P) - u$ is equal to $-\sigma_{i-1} \cdot \mathrm{sign}(P(u))$.*

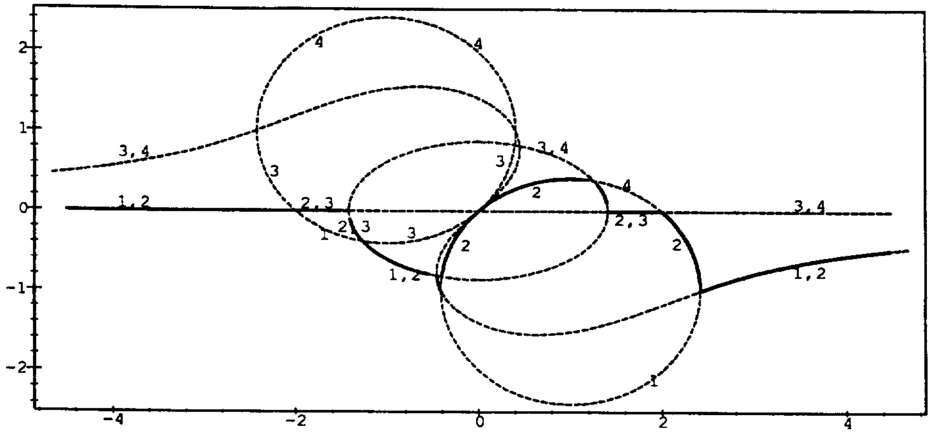

Fig. 1.

Proof. Part (a) is a direct consequence of the formal construction of the symbols ρ. Part (b) comes from (a) when $u = \rho_\sigma(P)$. If $u \neq \rho$ then the result is clear when the considered symbol ρ_σ corresponds to an actual Thom's root of P coded by σ. As any ρ_σ is an actual Thom's root of some derivative $P^{[j+1]}$ coded by $\sigma^{[j]}$ we compare u with $\rho_{\sigma^{[j]}}(P^{[j+1]})$ as in the previous case (details left to the reader). □

4. Examples

Example 4.1. Fig. 1 is the picture of the complete situation of the rth virtual roots $\rho_{4,j}(x) := \rho_{4,j}(P)(x)$ of the polynomial $P(x,y) = ((x-1)^2 + (y+1)^2 - 2)((x+1)^2 + (y-1)^2 - 2)$ considered as a polynomial in y parametrized by x. In Fig. 1 we can see the union of two circles corresponding to the zeroes of P, a cubic corresponding to the zeroes of P'_y, an ellipse corresponding to the zeroes of P''_y and the y-axis being the zero locus of $P_y^{(3)}$. The number j on the picture denotes $\rho_{4,j}(x)$ and $\rho_{4,2}(x)$ has been drawn in thick.

Example 4.2. Table 1 gives the complete situation of the ρ_σ upto degree 5, and is easy to extend to any degree.

Table 1

↓ degree of the polynomial derivative	
0	+
1	− • +
2	+ • −	− • +
3	− • +	+ • −	+ • −	− • +
4	+ • −	− • +	− • +	+ • −	− • +	+ • −	+ • −	− • +
5	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•

Every point • in the table denotes a function ρ_σ where the list σ is obtained reading from the top until the considered point •. If we want to add a line to the table, we do it in such a way that each sign of the bottom line subdivides in two, the first sign of the two being the opposite of the existing sign. In the previous table it is easy to find some evident incompatibilities:

- In degree 3 it is impossible to have the symbols $\rho_{[+,-,-]}$ and $\rho_{[+,+,-]}$ representing the real roots of a polynomial because we would have a polynomial with two consecutive simple roots giving the same sign to the derivative.
- In degree 4 we get two incompatibilities with the same type than the previous one, $\rho_{[+,-,-,+]}$ with $\rho_{[+,-,-,+]}$ and $\rho_{[+,+,-,-]}$ with $\rho_{[+,+,+,-]}$.
- Again in degree 4 a stronger new type of incompatibility appears: it is impossible to have simultaneously nonempty the two consecutive intervals $\mathrm{F}_{[+,-,-,-]}(P)$ and $\mathrm{F}_{[+,+,-,+]}(P)$. If $\mathrm{F}_{[+,-,-,-]}(P)$ and $\mathrm{F}_{[+,+,-,+]}(P)$ were nonempty then the polynomial P' would decrease from $-$ to $+$.
- If, for example, $\rho_{[+,-,-,-]}(P)$ is an actual Thom's root, then the interval $\mathrm{G}_{[+,+,-,+]}(P)$ is formed by only one point. Moreover, in this case, the roots coded by $[+,+,-,-]$ and $[+,+,+,-]$ cannot exist for P.

An exhaustive analysis of Table 1 allows to find, by similar arguments, all the possible simultaneous Thom's codings for the real roots of the same polynomial.

Example 4.3. Max and Min are rth root functions and Thom's root functions:

$$\max\{a_1,\ldots,a_k\} = \rho_{k,1}\left(\prod_{i=1}^{k}(x-a_i)\right) = \rho_{[+,+,+,\ldots,+]}\left(\prod_{i=1}^{k}(x-a_i)\right),$$

$$\min\{a_1,\ldots,a_k\} = \rho_{k,k}\left(\prod_{i=1}^{k}(x-a_i)\right) = \rho_{[+,-,+,-,\ldots]}\left(\prod_{i=1}^{k}(x-a_i)\right).$$

The nth root function can be described as

$$\sqrt[n]{\max(a,0)} = \rho_{n,n}(x^n-a) = \rho_{[+,+,+,\ldots,+]}(x^n-a).$$

Example 4.4 (*Root functions for a polynomial of degree* 3). We consider the polynomial $P = x^3 + 3px + 2q$. The complement of $pq(p^3+q^2) = 0$ in the plane (p,q) (Fig. 2) has six connected components, $\{A_i : 1 \le i \le 6\}$, obtained by giving strict signs to p, q and p^3+q^2. The border of these open sets will not be considered because the root functions extend there continuously.

Inside every A_i each of the four Thom's root functions has a fixed expression as an actual Thom's root of P or one of its derivatives. This fact is shown in the following

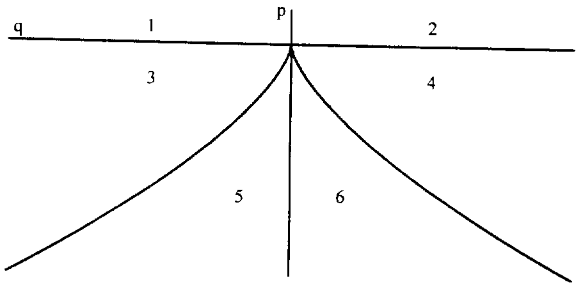

Fig. 2.

table:

$$\rho_{[+,+,+]}(P) = \begin{cases} \rho_{[+,+,+]}(P) = \text{ the biggest real root of } P \text{ if } (p,q) \in A_1 \cup A_3 \cup A_5 \cup A_6 \\ \rho_{[+]}(P^{[1]}) = 0 \quad \text{if } (p,q) \in A_2 \\ \rho_{[+,+]}(P^{[2]}) = \text{ the positive real root of } P' \text{ (i.e. } \sqrt{-p}) \text{ if } (p,q) \in A_4 \end{cases}$$

$$\rho_{[+,-,+]}(P) = \begin{cases} \rho_{[+,-,+]}(P) = \text{ the smallest real root of } P \text{ if } (p,q) \in A_2 \cup A_4 \cup A_5 \cup A_6 \\ \rho_{[+]}(P^{[1]}) = 0 \quad \text{if } (p,q) \in A_1 \\ \rho_{[+,-]}(P^{[2]}) = \text{ the negative real root of } P' \text{(i.e. } -\sqrt{-p}) \text{ if } (p,q) \in A_3 \end{cases}$$

$$\rho_{[+,+,-]}(P) = \begin{cases} \rho_{[+,+,-]}(P) = \text{ the intermediate real root of } P \text{ if } (p,q) \in A_6 \\ \rho_{[+]}(P^{[1]}) = 0 \quad \text{if } (p,q) \in A_1 \cup A_2 \cup A_3 \cup A_5 \\ \rho_{[+,+]}(P^{[2]}) = \text{ the positive real root of } P' \text{ (i.e. } \sqrt{-p}) \text{ if } (p,q) \in A_4 \end{cases}$$

$$\rho_{[+,-,-]}(P) = \begin{cases} \rho_{[+,-,-]}(P) = \text{ the intermediate real root of } P \text{ if } (p,q) \in A_5 \\ \rho_{[+]}(P^{[1]}) = 0 \text{ if } (p,q) \in A_1 \cup A_2 \cup A_4 \cup A_6 \\ \rho_{[+,-]}(P^{[2]}) = \text{ the negative real root of } P' \text{(i.e. } -\sqrt{-p}) \text{ if } (p,q) \in A_3 \end{cases}$$

5. Links and common properties

In this section we are going to examine the relationship between the two kinds of virtual roots, and the properties they share. A question that comes first in mind is the following: is it possible to express one set of virtual roots in terms of the other? Proposition 5.2 below shows that the ρ_σ can be expressed in terms of $\rho_{d,j}$, but the converse is not yet known. Let us start with a definition

Definition 5.1. Let σ be a list of length $d > 0$, always with $\sigma_0 = +$. We define $j(\sigma)$ as $1 + \sum_{i=0}^{d}(1 + \sigma_i\sigma_{i+1})/2$ (the number of "no sign change" in σ plus one). For instance $j([+,-,-]) = 2$ or $j([+,+,-,-,-]) = 4$.

Then we have

Proposition 5.2. *Let P be a degree d monic polynomial and σ a list of length d. Then,*

(1) *If $U_\sigma(P) \neq \emptyset$ then $U_\sigma(P) \subseteq U_{d,j(\sigma)}(P)$.*
(2) *If $\rho_\sigma(P)$ is an actual Thom's root, then $\rho_\sigma(P) = \rho_{d,j(\sigma)}(P)$.*
(3) *In general, for $\rho_\sigma(P)$, we have the following expression, which allows to express inductively the ρ_σ functions as* sup–inf *combinations of the $\rho_{d,j}$ functions:*

$$\rho_\sigma(P) = \max\{\tau_\sigma^-(P'), \ \min\{\tau_\sigma^+(P'), \ \rho_{d,j(\sigma)}(P)\}\}.$$

Proof. Let us prove (1) by induction on d. If $d = 1$, everything is easy. If $d > 1$, if $U_\sigma(P) \neq \emptyset$, it is also the case for $U_{\sigma^{(1)}}(P')$ and so $U_{\sigma^{(1)}}(P') \subseteq U_{d-1,j(\sigma^{(1)})}$ by induction. By Proposition 3.7, we have two cases to consider:

(a) $U_{\sigma^{(1)}}(P') = U_\sigma(P)$ and in that case there is no zero of P inside $U_\sigma(P)$ and in particular $\rho_{d,j(\sigma)}$ and $\rho_{d,j(\sigma)-1}$ are outside $U_\sigma(P)$ (otherwise they would be in $U_{d-1,j(\sigma^{(1)})}(P')$ and they would be zeroes of P inside $U_\sigma(P)$). So $U_\sigma(P) \subseteq U_{d,j(\sigma)}(P)$.

(b) $F_{\sigma^{(1)}}(P')$ is the union of two nonempty intervals $F_{\sigma^{(1)},-\sigma_{d-1}}(P) \cup F_{\sigma^{(1)},\sigma_{d-1}}(P)$, meaning the common endpoint is a zero of P inside $U_{d-1,j(\sigma^{(1)})}(P')$: it must be $\rho_{d,j(\sigma^{(1)})}(P)$. So the left interval $U_{\sigma^{(1)},-\sigma_{d-1}}(P)$ is contained in $U_{d,j(\sigma^{(1)})}(P) = U_{d,j(\sigma^{(1)},-\sigma_{d-1})}(P)$ and the right one $U_{\sigma^{(1)},\sigma_{d-1}}(P)$ is contained in $U_{d,1+j(\sigma^{(1)})}(P) = U_{d,j(\sigma^{(1)},\sigma_{d-1})}(P)$, which proves (1).

It is not hard to see that (1) implies (2): if $\rho_\sigma(P)$ is an actual Thom's root, then $U_{\sigma^{(1)}}(P')$ is not empty and is contained in $U_{d-1,j(\sigma^{(1)})}(P')$. The same is true for the closed corresponding intervals and the only zero of P in $F_{d-1,j(\sigma^{(1)})}(P')$ is then $\rho_{d,j(\sigma)}(P) = \rho_\sigma(P)$.

It is now easy to show (3): if $U_{\sigma^{(1)}}(P') = \emptyset$, then $G_{\sigma^{(1)}}(P')$ is a point and the formula in (3) for $\rho_\sigma(P)$ gives that point. If it is not empty, the same arguments as above show that, if $\rho_\sigma(P)$ is inside this open set, it must be equal to $\rho_{d,j(\sigma)}(P)$, and if it is an endpoint of $F_{\sigma^{(1)}}(P')$, then one of the intervals $U_{\sigma^{(1)},-\sigma_{d-1}}(P)$ or $U_{\sigma^{(1)},\sigma_{d-1}}(P)$ is not empty and so contained in the corresponding $U_{d,j(\sigma)}(P)$, σ being one of the two possible extensions of $\sigma^{(1)}$. But then the formula gives the end point of $F_{\sigma^{(1)}}(P')$ corresponding to $\rho_\sigma(P)$. Finally, use the remark following Definition 3.4 in order to replace in Proposition 5.2(3) $\tau_\sigma^\varepsilon(P')$ by some $\rho_{\sigma^{[j]}}(P^{[j+1]})$ (where $j < d-1$ depends only on σ and ε). □

We have already proved that the virtual roots are continuous in the coefficients of the polynomials, but we know a little more: on a given compact ball of $\mathbb{R}^d$, they are of course uniformly continuous and we can compute the modulus of continuity in terms of the radius of the ball. Let us start with a notation.

Notation 5.3. Let X be a complete metric space. We shall denote by $\mathrm{Ms}_k(X)$ the metric space of multisets with k elements in X, i.e. the complete metric space obtained from X^k with the semidistance

$$d_{\mathrm{Ms}}((x_i)_{i=1,\dots,k},(y_i)_{i=1,\dots,k}) \stackrel{\text{def}}{=} \min_{\lambda\in S(k)}\left\{\max_{i=1,\dots,k} d(x_i,y_{\lambda(i)})\right\}.$$

Then we need the following lemma.

Lemma 5.4. *Let U be a convex set in $\mathbb{R}^n$, X a metric space, $f:U\longrightarrow X$ a continuous function and $F:U\longrightarrow \mathrm{Ms_k}(X)$ a uniformly continuous function with modulus of uniform continuity $\omega(\varepsilon)$. Assume that for all $u\in U$, $f(u)\in F(u)$. Then f admits as modulus of uniform continuity the function: $\varepsilon\longmapsto\omega(\varepsilon/2k)$.*

Proof. We assume w.l.o.g. that U is the unit interval and $u=0$. We start with ε and search for δ such that for all $u'\in(0,\delta)$, we have $d(f(u),f(u'))<\varepsilon$. Let $\varepsilon'=\varepsilon/k$, $\delta=\omega(\varepsilon/2k)$ and $u'\in(0,\delta)$.

Either all $F(u)$ is in the open ball $B_X(f(u),(k-1)\varepsilon')$, and then $d(f(u),f(u'))<(k-1)\varepsilon'+\varepsilon/2k<\varepsilon$. Or there exists a $j<k-1$ such that $F(u)$ is contained in the disjoint union of the open ball $B_X(f(u),j\varepsilon')$ and of the complement $X-B_X(f(u),(j+1)\varepsilon')$. Then, for all $t\in[0,u']$, the set $F(t)$ is contained in the disjoint union of the open ball $B_X(f(u),j\varepsilon'+\varepsilon/2k)$ and of the complement of the corresponding closed ball $X-\overline{B}_X(f(u),(j+1)\varepsilon'-\varepsilon/2k)$. So, by connectivity, the point $f(t)$ must remain in the first of these two disjoint open sets and $d(f(u),f(u'))<j\varepsilon'+\varepsilon/2k<\varepsilon$. □

Then we have

Theorem 5.5 (Root functions local uniform continuity). *When the $|a_i|^{d-i}$ are bounded by $M\ge 1$, a modulus of uniform continuity $\omega(M,\varepsilon)$ (i.e. a function giving δ from ε in the definition of uniform continuity, with the l_1 norm in $\mathbb{R}^d$) for the functions $\rho_\sigma(a_{d-1},\dots,a_0)$ and $\rho_{d,j}(a_{d-1},\dots,a_0)$ is*

$$\omega(M,\varepsilon)=2M\left(\frac{\varepsilon}{d(d+1)(2d-1)M}\right)^d$$

Proof. A modulus of uniform continuity $\omega(M,\varepsilon)$ for the functions ρ_σ and $\rho_{d,j}$ is obtained using the following technical result appearing in [4, Appendix A, p. 276]:

The multiset root function for monic degree d polynomials:

$$\begin{aligned}\mathbb{C}^d &\longrightarrow \mathrm{Ms}_d(\mathbb{C})\\ P &\longmapsto \{\alpha\in\mathbb{C}:P(\alpha)=0\}\end{aligned}$$

(considering multiplicity) when the $|a_i|^{d-i}$ are bounded by $M\ge 1$, admits the following modulus of local uniform continuity, with the l_1 norm in $\mathbb{C}^d$:

$$2M\left(\frac{\varepsilon}{2M(2d-1)}\right)^d.$$

Applying the lemma with the Ostrowski modulus for the multiset union of complex roots of the polynomials P and its derivatives (the modulus of uniform continuity for the multiset of zeros of P is also good for its derivatives), we get the theorem. □

Remark that Ostrowski's bound and Lemma 5.4 allows to determine explicitly a modulus of local uniform continuity for any integral semi-algebraic continuous function by merely regarding the vanishing monic polynomial for the considered function.

6. Applications

In this section, we conclude with two applications. The first one is the following continuous version of Thom's lemma.

Theorem 6.1 (A continuous version for Thom's lemma). *Let d be an integer ≥ 1 and $\sigma=[\sigma_0,\dots,\sigma_d]$ a list of elements in $\{+,-\}$. We shall consider the monic polynomials with degree d as points of $\mathbb{R}^d$. If we define the sets of $\mathbb{R}^d$:*

$$W_\sigma=\{P\in\mathbb{R}^d : F_\sigma(P)\neq\emptyset\},\qquad V_\sigma=\{P\in\mathbb{R}^d : U_\sigma(P)\neq\emptyset\}$$

then the following statements are verified:

(1) *W_σ is a connected and closed $\mathbb{Q}$-semi-algebraic set whose interior is V_σ.*

(2) *V_σ is a connected and open $\mathbb{Q}$-semi-algebraic set whose closure is W_σ.*

(3) *For every P in W_σ the set $F_\sigma(P)$ is a nonempty closed interval and every finite end-point of $F_\sigma(P)$ is an integral continuous function of P and a root of P^*.*

(4) *Only two cases where an infinity end-point can appear:*

$$\sigma=[+,+,\dots,+]\longrightarrow+\infty,\qquad \sigma=[+,-,+,-,+,\dots]\longrightarrow-\infty.$$

Proof. Parts (3) and (4) are clear after the detailed study on the sets $F_\sigma(P)$ made in the previous sections. The following equivalences:

$$F_\sigma(P)\neq\emptyset \iff \tau_\sigma^+(P)\in F_\sigma(P),\qquad U_\sigma(P)\neq\emptyset \iff \frac{\tau_\sigma^+(P)+\tau_\sigma^-(P)}{2}\in U_\sigma(P)$$

allow to show that W_σ is a closed $\mathbb{Q}$-semi-algebraic set and that V_σ is an open $\mathbb{Q}$-semi-algebraic set.

Now we suppose w.l.o.g. that $\sigma_d=\sigma_{d-1}=+$ and that we are not in an infinity case. For a degree $d-1$ polynomial R we define

$$R_1(x)=d\int_0^x R(t)\,\mathrm{d}t,\qquad \psi(R)=R_1(\tau^+_{\sigma^{(1)}}(R)).$$

A simple verification provides the following description for the sets W_σ and V_σ:

$$W_\sigma=\{P : F_{\sigma^{(1)}}(P')\neq\emptyset,\ \psi(P'/d)\geq -P(0)\}=W_{\sigma^{(1)}}\times\mathbf{R}\cap\{P:\psi\circ\pi(P)\geq -P(0)\},$$

$$V_\sigma=\{P : U_{\sigma^{(1)}}(P')\neq\emptyset,\ \psi(P'/d)> -P(0)\}=V_{\sigma^{(1)}}\times\mathbf{R}\cap\{P:\psi\circ\pi(P)> -P(0)\},$$

where π is the projection:

$$\pi:\ \mathbb{R}^d \longrightarrow \mathbb{R}^{d-1},$$
$$P \longmapsto P'/d.$$

Proceeding by induction on d we obtain the remaining claims in (1) and (2) because W_σ and V_σ are cylinders bounded from below by the continuous semi-algebraic function $\psi \circ \pi + P(0)$ and whose base is a semi-algebraic set verifying the conditions in (1) and (2) by induction hypothesis. □

The second question we want to address here is the following: what kind of functions do we get if we take the closure under inf–sup of the set of functions ρ_σ or $\rho_{d,j}$? If we take a given continuous function on $\mathbb{R}^n$ which is integral over the n variable polynomials $\mathbb{R}[X_1,\dots,X_n]$, it is annihilated by a monic polynomial $Q(\underline{X},Y)$ in Y with coefficients in $\mathbb{R}[X_1,\dots,X_n]$, and piecewise on $\mathbb{R}^n$, it is a precise real root of $Q(\underline{X},Y)$ (in terms of rth roots or Thom's roots), but in general, it does not admit a global description as Inf–Sup of the virtual roots of $Q(\underline{X},Y)$. A very simple example is the following:

Example 6.2. Take $Q(X,Y)=Y^2-X^2$, and $f(X)=X$. If we had a description of f as Inf–Sup of virtual roots of Q, it would depend only on X^2, and so would be the same for $X>0$ and $X<0$. Of course, we have other nice descriptions for f! But it means that if we want to describe integral continuous functions as Inf–Sup of virtual roots, we have to use other polynomials than Q. Theorem 6.4 discusses this aspect.

Definition 6.3. If ρ is either a rth root or Thom's root function on $\mathbb{R}^d$, we define functions on $\mathbb{R}^n$ in filling each occurrence of ρ with a polynomial in n variables. Let us call "polyroots in n variables" these functions on $\mathbb{R}^n$ (in both cases), and "Inf–Sup of polyroots" the functions obtained in taking finite infima and suprema of such functions.

Then we get the following:

Theorem 6.4. *The closure of polyroots in h variables under sum, Inf and Sup (in both cases of polyroots) is the integral closure of* $\mathbb{R}[X_1,\dots,X_h]$ *in the ring of continuous functions on* $\mathbb{R}^h$.

Proof. It is clear that the Inf–Sup of sums of polyroots in h variables are continuous and integral over the polynomial ring $\mathbb{R}[X_1,\dots,X_h]$, so the only thing to prove is the converse. Let $f:\mathbb{R}^h \longrightarrow \mathbb{R}$ be an integral continuous function and $Q(x_1,\dots,x_h,y)$ a polynomial in $\mathbb{R}[x_1,\dots,x_h,y]$, y–monic, with degree d in y, and verifying:

$$Q(\alpha_1,\dots,\alpha_h,\ f(\alpha_1,\dots,\alpha_h))=0 \quad \forall(\alpha_1,\dots,\alpha_h)\in\mathbb{R}^h.$$

We shall denote $\underline{x} = (x_1, \dots, x_h)$ and write:

$$Q(\underline{x}, y) = y^d + \sum_{k=0}^{d-1} Q_k(\underline{x}) y^k.$$

Let $g_1, \dots, g_m$ be the virtual root functions corresponding to degree d ($m = d$ in case of $\rho_{d,j}$ and $m = 2^{d-1}$ in case of Thom's roots). Then for every $i \in \{1, \dots, m\}$ the function defined by

$$l_i(\underline{x}) = g_i(Q_{d-1}(\underline{x}), \dots, Q_0(\underline{x}))$$

is a polyroot. After these definitions it is clear that the function

$$\prod_{i=1}^{m} (f(\underline{x}) - l_i(\underline{x}))$$

is zero everywhere.

Next, for every $i \in \{1, \dots, m\}$, we introduce the closed semi-algebraic set:

$$F_i = \{(\alpha_1, \dots, \alpha_h) \in \mathbb{R}^h : f(\alpha_1, \dots, \alpha_h) = l_i(\alpha_1, \dots, \alpha_h)\}$$

whose interior will be denoted by U_i.

Applying the Finiteness theorem we describe every U_i as a finite union of basic semi-algebraic open sets, i.e. by strict sign conditions over polynomials in $\mathbb{R}[x_1, \dots, x_h]$. Let $\{P_j : j \in J\}$ be the family of polynomials appearing in such a description and $\{P_j : j \in K\}$ the family obtained by completing the previous one until obtaining a separating family.

Finally, we consider the nonempty open sets obtained in giving strict signs to the polynomials in $\{P_j : j \in K\}$. This family will be denoted by $\{V_n : n \in N\}$. As our family of polynomials is separating then the closed semi-algebraic set obtained replacing in the description for V_n the condition $<$ by $\leq$ and the condition $>$ by $\geq$ is the closure of V_n. Moreover, after the definition of the V_n's it is clear that they are disjoint: $n \neq p \iff V_n \cap V_p = \emptyset$.

The conclusion of the theorem will be obtained in constructing a sum of Inf–Sup of polyroots equal to f over the union of the sets V_n (which is dense in $\mathbb{R}^h$).

For every $n \in N$ let i_n be such that $V_n \subseteq U_{i_n}$: this implies that the function f, over V_n, is equal to l_{i_n}. Now we construct for every pair (n, p) with $n \neq p$ an Inf–Sup of polyroots $v_{n,p}$ verifying the following conditions:

$$\forall \underline{\alpha} \in V_n \quad v_{n,p}(\underline{\alpha}) \geq f(\underline{\alpha}) = l_{i_n}(\underline{\alpha}),$$
$$\forall \underline{\alpha} \in V_p \quad v_{n,p}(\underline{\alpha}) \leq f(\underline{\alpha}) = l_{i_p}(\underline{\alpha}).$$

If $i_n = i_p$ we define $v_{n,p} = l_{i_n}$. So, without loss of generality, we can assume that $(n, p) = (1,2)$, $f = l_1$ on V_1 and $f = l_2$ on V_2. Let W_1 and W_2 be the closures of

V_1 and V_2 and write (w.l.o.g.):

$$V_1 = \{\underline{\alpha} \in \mathbb{R}^h : P_1(\underline{\alpha}) > 0, \ldots, P_r(\underline{\alpha}) > 0, \ldots, P_s(\underline{\alpha}) > 0\},$$
$$V_2 = \{\underline{\alpha} \in \mathbb{R}^h : P_1(\underline{\alpha}) > 0, \ldots, P_r(\underline{\alpha}) > 0, P_{r+1}(\underline{\alpha}) < 0, \ldots, P_s(\underline{\alpha}) < 0\}.$$

This allows to derive the following descriptions for W_1 and W_2:

$$W_1 = \{\underline{\alpha} \in \mathbb{R}^h : P_1(\underline{\alpha}) \geq 0, \ldots, P_r(\underline{\alpha}) \geq 0, \ldots, P_s(\underline{\alpha}) \geq 0\},$$
$$W_2 = \{\underline{\alpha} \in \mathbb{R}^h : P_1(\underline{\alpha}) \geq 0, \ldots, P_r(\underline{\alpha}) \geq 0, P_{r+1}(\underline{\alpha}) \leq 0, \ldots, P_s(\underline{\alpha}) \leq 0\}.$$

Now we consider the polynomial:

$$R(\underline{x}) = \sum_{i=r+1}^{s} P_i(\underline{x}).$$

The description of W as union of W_1 and W_2 allows to conclude that inside W an equation for W_1 is $R(\underline{x}) \geq 0$ and the equation for W_2 is $R(\underline{x}) \leq 0$:

$$W_1 = \{\underline{\alpha} \in W : R(\underline{\alpha}) \geq 0\}, \qquad W_2 = \{\underline{\alpha} \in W : R(\underline{\alpha}) \leq 0\},$$

which implies the following description for $W_1 \cap W_2$:

$$W_1 \cap W_2 = \{\underline{\alpha} \in W : R(\underline{\alpha}) = 0\}.$$

On $W_1 \cap W_2$ we have $f = l_1 = l_2$ and every zero of $R(\underline{x})$ in W is a zero of $l_1(\underline{x}) - l_2(\underline{x})$. So applying Łojasiewicz inequality we obtain the existence of positive integers t and k, and a positive number $c \in \mathbb{R}$ verifying:

$$|l_1(\underline{\alpha}) - l_2(\underline{\alpha})|^t \leq c|R(\underline{\alpha})|(1 + \|\underline{\alpha}\|^2)^k \qquad \forall \underline{\alpha} \in W.$$

This allows to define the function:

$$v_{1,2}(\underline{\alpha}) = l_2(\underline{\alpha}) + \sqrt[t]{\max\{0, cR(\underline{\alpha})(1 + \|\underline{\alpha}\|^2)^k\}}$$

verifying the desired conditions:

- for all $\underline{\alpha} \in W_2$ we have $v_{1,2}(\underline{\alpha}) = l_2(\underline{\alpha})$,
- for all $\underline{\alpha} \in W_1$ we have:

$$v_{1,2}(\underline{\alpha}) \geq l_2(\underline{\alpha}) + |l_1(\underline{\alpha}) - l_2(\underline{\alpha})| \geq l_1(\underline{\alpha}).$$

Once all the functions $v_{n,p}$ have been constructed, it is very easy to check that

$$f(\underline{\alpha}) = \min\{\max\{v_{n,p}(\underline{\alpha}) : n \neq p,\ n \in N\} : p \in N\}$$

and the proof of the theorem is obtained. □

References

[1] J. Bochnak, M. Coste and M.F. Roy, Géométrie Algébrique Réelle, Ergebnisse, Vol. 12 (Springer, Berlin, 1987).

[2] C.N. Delzell, On the Pierce–Birkhoff conjecture over ordered fields, Rocky Mountain J. Math. 19 (1989) 651–668.

[3] L. Mahé, On the Pierce–Birkhoff conjecture, Rocky Mountain J. Math 14 (1984) 983–985.

[4] A.M. Ostrowski, Solutions of Equations and Systems of Equations (Academic Press, New York, 1966).